On certain permutation representations of the braid group

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Abstract

In this paper we present a structural theorem, concerning certain homomorphic images of Artin braid group on n strands in finite symmetric groups. It is shown that each one of these permutation groups is an extension of the symmetric group S_n by an appropriate abelian group, and in "half" of the cases this extension splits.

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1 Introducton

In [1], E. Artin studies the permutation representations of his braid group B_n in the symmetric group S_n as a first step toward the determination of all automorphisms of B_n. He considers the last problem to be "quite difficult" and notes: "Before one can attack it, one has obviously first to get the permutations aut of the way."

The aim of this paper is to present the proof of a structure theorem concerning certain permutation representations of B_n. The corresponding finite permutation groups are defined as subgroups of a wreath product, and this allows their description as an (in "half" of the cases, split) extension of the symmetric group S_n by an abelian group.

The text is organized as follows. In the beginning of Section 2 we remind some terminology and simple statements, concerning wreath products of the type $W \wr S_n$, where $W \leq S_d$ is a permutation group, and use a special endomorphism ω of the symmetric group S_∞ in order to simplify the notation. The permutations σ from the non-trivial coset of the base group $W^{(2)}$ of $W \wr S_2$ play a crucial role in this paper. If we consider such a σ, and its translate $\omega^d(\sigma)$, Theorem 3 establishes four conditions that are equivalent to the fact that this

pair of permutations is braid-like. As a by-product we obtain four necessary and sufficient conditions for two permutations in S_d to commute in terms of (the cycle structure of) their common product. In particular, one of these conditions gives explicit expressions for the two commuting permutations, see Remark 5.

Section 3 is devoted to the description of certain n-braid-like groups $B_n(\sigma)$ (that is, homomorphic images of B_n) which are subgroups of some finite symmetric group. These groups are generated by the braid-like couples σ, $\omega^d(\sigma)$, and several of their ω^d-translates. The permutation σ depends on a fixed permutation $\tau \in S_d$, and on a permutation u of the set $\mathcal{C}(\tau)$ of cycles of τ, see 2. In Proposition 6 we show that, in general, the group $B_n(\sigma) \leq S_d \wr S_n$ is intransitive, find its sets of transitivity $(Y_o)_{o\in\mathcal{C}(\tau)}$, as well as the restrictions $B_n(\sigma^{(o)})$ of $B_n(\sigma)$ on Y_o, which are groups of the same type. Moreover, $B_n(\sigma)$ is a subdirect product of $B_n(\sigma^{(o)})$'s. In particular, the group $B_n(\sigma)$ is transitive if and only if the permutation u is a long cycle. Let q be the order of the permutation τ. Theorem 9 is the central result of the paper, where we prove that each n-braid-like group $B_n(\sigma)$ is an extension of the symmetric group S_n by an abelian group. If we assume, in addition, that q is odd, then this extension splits, and we can use the method of "little groups" of Wigner and Mackey (see [4, Proposition 25]) for finding, in principle, of all finite-dimensional irreducible representations of $B_n(\sigma)$. Thus, we can obtain a series of finite-dimensional irreducible representations of Artin braid group B_n.

2 Braid-like pairs of permutations

First, we introduce some notation. We define the symmetric group S_∞ as the group of all permutations of the set of positive integers, which fix all but finitely many elements. The symmetric group S_d is identified with the subgroup of S_∞, consisting of all permutations fixing any $k > d$. The conjugation by a permutation ζ in S_∞ is denoted by c_ζ.

From now on we assume that d and n are integers, $d \geq 2$, $n \geq 3$. Given a permutation $\zeta \in S_d$, we denote by $\varrho(\zeta) = (1^{c_1(\zeta)}, 2^{c_2(\zeta)}, \dots, d^{c_d(\zeta)})$ its cycle type (here $c_i(\zeta)$ is the number of cycles of ζ of length i). For any partition λ of d, $\lambda = (1^{m_1}, 2^{m_2}, \dots, d^{m_d})$, we set $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots d^{m_d} m_d!$.

We denote by ω the injective endomorphism of S_∞, defined via the rule

$$(\omega(\sigma))(k) = \sigma(k-1) + 1,\ k \geq 2,\ (\omega(\sigma))(1) = 1.$$

As usual, we identify the wreath product $S_d \wr S_n$ with the image of its natural faithful permutation representation, see [2, 4.1.18], and for each subgroup $W \leq S_d$ we identify the wreath product $W \wr S_n$ with its image via the above inclusion.

Let $\theta_s \in S_{nd}$, be the involutions

$$\theta_s = \begin{pmatrix} (s-1)d+1 & \cdots & sd & sd+1 & \cdots & (s+1)d \\ sd+1 & \cdots & (s+1)d & (s-1)d+1 & \cdots & sd \end{pmatrix} \in S_{nd},$$

$s = 1, \dots, n-1$. We set $\Sigma_n = \langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle \leq S_{nd}$. Then the direct product $W^{(n)} = W\omega^d(W) \dots \omega^{(n-1)d}(W)$ is a normal subgroup of of the wreath product

$W \wr S_n$ (its base group), Σ_n is a complement of $W^{(n)}$, and we have that $W \wr S_n$ is the semidirect product of Σ_n by $W^{(n)}$: $W \wr S_n = W^{(n)} \cdot \Sigma_n$. The isomorphism $\Sigma_n \simeq S_n$ maps the involution θ_s onto the transposition $(s, s+1)$, $s = 1, \dots n-1$.

In particular, for any $W \leq S_d$ the wreath product $W \wr S_2$ is the semidirect product of $\Sigma_2 = \langle \theta \rangle$ by its base group $W^{(2)}$, where $\theta = \theta_1$. The left coset $\theta W^{(2)}$ of $W^{(2)}$ in $W \wr S_2$ consists of permutations of $[1, 2d]$ that map $[1, d]$ onto $[d+1, 2d]$.

Let ζ be a permutation of a finite set. By $\mathcal{C}_m(\zeta)$ we denote the set of all cycles of length m of ζ, and set $\mathcal{C}(\zeta) = \cup_{m \geq 1} \mathcal{C}_m(\zeta)$. Let us fix a permutation $\tau \in W$. We denote by $M_{W,\tau}$ the set of all permutations $\sigma \in \theta W^{(2)}$ that satisfy the equation

$$\sigma^2 = \tau \omega^d(\tau), \tag{1}$$

and let $M_W = \cup_{\tau \in S_d} M_{W,\tau}$. We set $M_{d,\tau} = M_{S_d,\tau}$, and $M_d = M_{S_d}$. Let us denote by $N_{d,\tau}$ the set of all permutations $\sigma \in S_d \wr S_2$, constructed in the following way:

(A) fix a permutation u of the set $\mathcal{C}(\tau)$, that maps the subset $\mathcal{C}_m(\tau)$ onto itself for any $m = 1, \dots, d$;

(B) choose an $m = 1, \dots, d$;

(C) for any $\alpha \in \mathcal{C}_m(\tau)$ choose an initial element i_1 of α, an initial element j_1 of $u_m(\alpha)$, and write α and $u(\alpha)$, using the cycle notation:

$$\alpha = (i_1, i_2, \dots, i_m),$$

$$u(\alpha) = (j_1, j_2, \dots, j_m);$$

(D) shuffle the cycle notations of α and $\omega^d(u(\alpha))$ from (C), and get a cycle of length $2m$:

$$(i_1, j_1 + d, i_2, j_2 + d, \dots, i_m, j_m + d); \tag{2}$$

(E) multiply the cycles (2) for all $\alpha \in \mathcal{C}(\tau)$, and denote the permutation thus obtained by σ.

If in the above construction sequence (A), (B), (C), (D), (E), we replace the steps (D), (E) with the steps (D′), (E′), (F′) below, the result is a binary relation $R_{d,\tau}$ on S_d. The set $R_{d,\tau}$ consists of all ordered pairs $(\varsigma_1, \varsigma_2) \in S_d \times S_d$, obtained by the construction sequence (A), (B), (C), (D′), (E′), (F′), where

(D′) denote by p_α the bijection

$$supp(\alpha) \to supp(u(\alpha)),\ i_1 \mapsto j_1, \dots, i_m \mapsto j_m,$$

and by q_α the bijection

$$supp(u(\alpha)) \to supp(\alpha),\ j_1 \mapsto i_2, \dots, j_{m-1} \mapsto i_m, j_m \mapsto i_1,$$

in case $m \geq 2$, and $p_\alpha = q_\alpha = (i_1, j_1)$ in case $m = 1$;

(E′) choose a cycle o of the permutation u, and set

$$\varsigma_1^{(o)} = \coprod_{\alpha \in o} p_\alpha \in S_{X_o}, \text{and } \varsigma_2^{(o)} = \coprod_{\alpha \in o} q_\alpha \in S_{X_o},$$

where $X_o = \cup_{\alpha \in o} supp(\alpha)$;

(F′) set

$$\varsigma_1 = \prod_{o \in \mathcal{C}(u)} \varsigma_1^{(o)} \in S_d, \text{and } \varsigma_2 = \prod_{o \in \mathcal{C}(u)} \varsigma_2^{(o)} \in S_d.$$

Let $R_d = \cup_{\tau \in S_d} R_{d,\tau}$, and $N_d = \cup_{\tau \in S_d} N_{d,\tau}$.

The family $(X_o)_{o \in \mathcal{C}(u)}$ of sets is a partition of the integer-valued interval $[1, d]$. Let θ_o be the restriction of the permutation θ on $X_o \cup X_o + d$. Let us set $\tau^{(o)} = \prod_{\alpha \in o} \alpha$, so $\tau = \prod_{o \in \mathcal{C}(u)} \tau^{(o)}$. Note that each one of the families

$$(\varsigma_1^{(o)})_{o \in \mathcal{C}(u)},\ (\varsigma_2^{(o)})_{o \in \mathcal{C}(u)},\ (\theta_o)_{o \in \mathcal{C}(u)},\ \text{and } (\tau^{(o)})_{o \in \mathcal{C}(u)},$$

consists of permutations with pairwise disjoint supports.

Next technical lemma can be proved by inspection.

Lemma 1 *(i) Let $o = (\alpha, u(\alpha), u^2(\alpha), \ldots)$ be a cycle of the permutation u, and let us choose cyclic notations for the members of o: $\alpha = (i_1, i_2, \ldots, i_m)$, $u(\alpha) = (j_1, j_2, \ldots, j_m)$, $u^2(\alpha) = (k_1, k_2, \ldots, k_m)$,.... Then*

$$\theta_o \varsigma_1^{(o)} \omega^d (\varsigma_2^{(o)}) =$$

$$(i_1, j_1 + d, i_2, j_2 + d, \ldots, i_m, j_m + d)(j_1, k_1 + d, j_2, k_2 + d, \ldots, j_m, k_m + d) \ldots;$$

(ii) $\varsigma_1^{(o)} \varsigma_2^{(o)} = \varsigma_2^{(o)} \varsigma_1^{(o)} = \tau^{(o)}$;

(iii) if $\sigma \in N_{d,\tau}$, then the ordered pair $(\varsigma_1, \varsigma_2) \in R_{d,\tau}$, obtained by the construction sequence with the same initial segment, is such that $\sigma = \theta \varsigma_1 \omega^d(\varsigma_2)$;

(iv) if $(\varsigma_1, \varsigma_2) \in R_{d,\tau}$, then $\varsigma_1 \varsigma_2 = \varsigma_2 \varsigma_1 = \tau$.

The action of a group on itself via conjugation can be used in order to prove

Lemma 2 *Let $W \leq S_d$ be a permutation group, and let $p(W)$ be the number of conjugacy classes of the group W. Then the number of ordered pairs of elements of W, whose components commute, is $|W|p(W)$.*

A pair of permutations η, ζ from S_∞ is said to be *braid-like* if $\eta\zeta \neq \zeta\eta$, and $\eta\zeta\eta = \zeta\eta\zeta$.

Theorem 3 *Let $W \leq S_d$ be a permutation group, and let $\sigma \in \theta W^{(2)}$, $\sigma = \theta \varsigma_1 \omega^d(\varsigma_2)$, where $\varsigma_1, \varsigma_2 \in W$. The following three statements are then equivalent:*

(i) the pair of permutations σ, $\omega^d(\sigma)$ is braid-like;

(ii) one has $\sigma \in M_W$;

(iii) the permutations ς_1 and ς_2 commute.

Under these conditions, if $\sigma \in M_{W,\tau}$, $\tau \in W$, then $\varsigma_1 \varsigma_2 = \varsigma_2 \varsigma_1 = \tau$. If, in addition, $W = S_d$, then parts (i) – (iii) are equivalent to each one of

(iv) one has $\sigma \in N_d$;

(v) one has $(\varsigma_1, \varsigma_2) \in R_d$.

Proof: Suppose that (i) holds, and let $\eta \in S_d$ be the permutation with $\sigma(i) = d + \eta(i)$ for any $i \in [1, d]$. We have

$$\sigma\omega^d(\sigma)\sigma(i) = \sigma\omega^d(\sigma)(d + \eta(i)) = d + \sigma\eta(i),$$

and

$$\omega^d(\sigma)\sigma\omega^d(\sigma)(i) = \omega^d(\sigma)\sigma(i) = \omega^d(\sigma)(d + \eta(i)) = d + \sigma\eta(i),$$

for $i \in [1, d]$. Let us set $i(j) = j - d$ where $j \in [d + 1, 2d]$. We have

$$\sigma\omega^d(\sigma)\sigma(j) = \sigma^2(j) = \sigma^2(d + i(j)),$$

and

$$\omega^d(\sigma)\sigma\omega^d(\sigma)(j) = \omega^d(\sigma)\sigma\omega^d(\sigma)(d + i(j)) = \omega^d(\sigma)\sigma(d + \sigma(i(j))) =$$

$$\omega^d(\sigma)(d + \sigma(i(j))) = d + \sigma^2(i(j)),$$

for any $j \in [d+1, 2d]$. Therefore the pair of permutations σ, $\omega^d(\sigma)$ is braid-like if and only if $\sigma^2(d+i) = d+\sigma^2(i)$ for any $i \in [1, d]$. Thus, part (i) is equivalent to part (ii). A comparison of the equalities (1), and $\sigma^2 = \varsigma_2\varsigma_1\omega^d(\varsigma_1\varsigma_2)$, yields the equivalence of parts (ii) and (iii).

Now, suppose $W = S_d$. In accord with Theorem 3, (ii), (iii), and Lemma 2, we obtain $|M_d| = p(d)d!$, where $p(d)$ is the number of partitions of d. On the other hand, let for any $\upsilon \in S_d$, $C(\upsilon)$ be its conjugacy class, and let C be a set of representatives of the conjugacy classes of S_d. We have

$$|N_d| = \sum_{\tau\in S_d} |N_{d,\tau}| = \sum_{\upsilon\in C} \sum_{\varsigma\in C(\upsilon)} |N_{d,\varsigma}| = \sum_{\upsilon\in C} \frac{d!}{z_{\varrho(\upsilon)}} z_{\varrho(\upsilon)} = p(d)d!.$$

Since for any $\tau \in S_d$ we have $N_{d,\tau} \subset M_{d,\tau}$, then $N_d \subset M_d$. Therefore, $N_{d,\tau} = M_{d,\tau}$ for all $\tau \in S_d$, and, in particular, parts (ii) and (iv) are equivalent.

Lemma 1, (iii), and (iv), yield that part (iv) implies part (v), and part (v) implies part (iii), respectively.

Corollary 4 *If $\sigma \in M_{W,\tau}$ for some $\tau \in W$, and $a \in \langle\tau\rangle^{(2)}$, $a = \tau^k\omega^d(\tau^\ell)$, then $\sigma \in M_{W,\tau^{k+\ell+1}}$.*

Proof: Let $\sigma = \theta\varsigma_1\omega^d(\varsigma_2)$, where $\varsigma_1, \varsigma_2 \in W$ commute, and $\tau = \varsigma_1\varsigma_2$. We have

$$\sigma a = \theta\varsigma_1\omega^d(\varsigma_2)\tau^k\omega^d(\tau^\ell) = \theta\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell),$$

and therefore

$$(\sigma a)^2 = \theta\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell)\theta\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell) = c_\theta(\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell))\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell) =$$

$$\omega^d(\varsigma_1\tau^k)\varsigma_2\tau^\ell\varsigma_1\tau^k\omega^d(\varsigma_2\tau^\ell) = \varsigma_2\tau^\ell\varsigma_1\tau^k\omega^d(\varsigma_1\tau^k\varsigma_2\tau^\ell) = \tau^{k+\ell+1}\omega^d(\tau^{k+\ell+1}).$$

Remark 5 *The equivalence of parts (iii) and (v) of Theorem 3 gives explicit expressions of any two permutations in the symmetric group S_d, which commute, in terms of the cyclic structure of their common product.*

3 Braid-like permutation groups

We remind the definition of Artin braid group B_n on n strands as an abstract group: this is the group generated by $n-1$ generators β_1, β_2, ..., β_{n-1}, subject to the following braid relations

$$\beta_r\beta_s = \beta_s\beta_r,\ |r-s| \geq 2,$$

$$\beta_r\beta_s\beta_r = \beta_s\beta_r\beta_s,\ |r-s| = 1.$$

A group is said to be *n-braid-like* if it is a homomorphic image of Artin braid group B_n.

Let $\sigma \in \theta(S_d)^{(2)}$, where $\theta = \theta_1 \in \Sigma_n$. The permutations

$$\sigma_1 = \sigma,\ \sigma_2 = \omega^d(\sigma), \ldots, \sigma_{n-1} = \omega^{(n-2)d}(\sigma)$$

are from the wreath product $S_d \wr S_n$; let $B_n(\sigma)$ be the subgroup of $S_d \wr S_n$, generated by them:

$$B_n(\sigma) = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle.$$

Let $W \leq S_d$ be a permutation group. We note that if $\sigma \in \theta W^{(2)}$, then $B_n(\sigma)$ is a subgroup of the wreath product $W \wr S_n$.

From now on let us suppose that the pair of permutations σ, $\omega^d(\sigma)$, where $\sigma \in \theta S_d \omega^d(S_d)$, is braid-like, and let $\sigma^2 = \tau\omega^d(\tau)$, $\tau \in S_d$. Using Lemma 1, (i), and Theorem 3, we have $\sigma = \prod_{o\in\mathcal{C}(u)} \sigma^{(o)}$, where $\sigma^{(o)} = \theta_o \varsigma_1^{(o)} \omega^d(\varsigma_2^{(o)})$. According to Lemma 1, (ii), and Theorem 3, (i), (iii), applied for $W = S_{X_o}$, we obtain that the pair of permutations $\sigma^{(o)} \in \theta_o(S_{X_o})^{(2)}$, $\omega^d(\sigma^{(o)})$, is braid-like. We set

$$\sigma_1^{(o)} = \sigma^{(o)},\ \sigma_2^{(o)} = \omega^d(\sigma^{(o)}), \ldots, \sigma_{n-1}^{(o)} = \omega^{(n-2)d}(\sigma^{(o)}),$$

$$B_n(\sigma^{(o)}) = \langle \sigma_1^{(o)}, \sigma_2^{(o)}, \ldots, \sigma_{n-1}^{(o)} \rangle,$$

and

$$Y_o = X_o \cup X_o + d \cup \ldots \cup X_o + (n-1)d,\ o \in \mathcal{C}\,(u)\,.$$

The family of sets $(Y_o)_{o\in\mathcal{C}(u)}$ is a partition of the integer-valued interval $[1, nd]$.

Proposition 6 *For any cycle o of the permutation u, one has:*

(i) the permutations $\sigma^{(o)} = \sigma_1^{(o)}$, $\sigma_2^{(o)}$, ..., $\sigma_{n-1}^{(o)}$ (respectively, the permutations $\sigma = \sigma_1$, σ_2, ..., σ_{n-1}) satisfy the braid relations;

(ii) the group $B_n(\sigma^{(o)})$ (respectively, the group $B_n(\sigma)$) is n-braid-like with surjective homomorphism $B_n \to B_n(\sigma^{(o)})$, (respectively, $B_n \to B_n(\sigma)$) defined by the map $\beta_s \mapsto \sigma_s^{(o)}, s = 1, \ldots, n-1$ (respectively, $\beta_s \mapsto \sigma_s, s = 1, \ldots, n-1$);

(iii) the group $B_n(\sigma^{(o)})$ is transitive on the set Y_o, the family $(Y_o)_{o\in\mathcal{C}(u)}$ consists of all sets of transitivity for $B_n(\sigma)$, and $B_n(\sigma)$ is a subdirect product of $B_n(\sigma^{(o)})$, $o \in \mathcal{C}\,(u)$: $B_n(\sigma) \leq \prod_{o\in\mathcal{C}(u)} B_n(\sigma^{(o)})$.

Proof: It is enough to prove parts (i) and (ii) for the permutations $\sigma^{(o)}$.

(i) If $|r-s| \geq 2$, then the supports of the permutations $\sigma_r^{(o)}$ and $\sigma_s^{(o)}$ are disjoint, so they commute. In case $|r-s|=1$ part (i) holds because the pair of permutations $\sigma_1^{(o)}$, $\sigma_2^{(o)}$, is braid-like.

(ii) Part (i) and [3, Lemma 1.2] yield immediately part (ii).

(iii) Straightforward inspection.

Part (iii) of the above theorem yields immediately

Corollary 7 *The group $B_n(\sigma)$ is transitive if and only if the permutation u of the set $\mathcal{C}(\tau)$ is a long cycle.*

The intersection $BW_n(\sigma) = B_n(\sigma) \cap W^{(n)}$ is a normal subgroup of $B_n(\sigma)$. In particular, if $\tau \in W$, and $\langle\tau\rangle$ is the cyclic group generated by τ, then $A_n(\sigma) = B_n(\sigma) \cap \langle\tau\rangle^{(n)}$ is an abelian normal subgroup of both $B_n(\sigma)$ and $BW_n(\sigma)$.

Lemma 8 *One has $\sigma_s^2 \in A_n(\sigma)$, the restriction of the conjugations c_{σ_s} on $A_n(\sigma)$ are involutions, and the following equalities hold:*

$$c_{\sigma_s}(\omega^{sd}(\tau^2)) = \omega^{(s-1)d}(\tau^2),$$

for $s = 1, \dots, n-1$, and

$$c_{\sigma_{r+1}}(\sigma_r^2) = c_{\sigma_r}(\sigma_{r+1}^2) = \omega^{r-1}(\tau)\omega^{(r+1)d}(\tau),$$

for $r = 1, \dots, n-2$.

Proof: Let $\sigma_1 = \theta\varsigma_1\omega^d(\varsigma_2)$. Theorem 3 yields $\varsigma_1\varsigma_2 = \varsigma_2\varsigma_1 = \tau$, and $\sigma_1^2 = \tau\omega^d(\tau) \in A_n(\sigma)$. Since the group $A_n(\sigma)$ is abelian, the conjugations $c_{\sigma_s^2}$ coincide with the identity of $A_n(\sigma)$, that is, $c_{\sigma_s} = c_{\sigma_s^{-1}}$ on the group $A_n(\sigma)$. It is enough to prove the equalities for $s = 1$ and $r = 1$. We have

$$c_{\sigma_1}(\omega^d(\tau^2)) = c_{\varsigma_2}(\tau^2)\omega^{2d}(1) = \tau^2,$$

$$c_{\sigma_2}(\sigma_1^2) = \tau\omega^{2d}(c_{\varsigma_1}(\tau)) = \tau\omega^{2d}(\tau),$$

$$c_{\sigma_1}(\sigma_2^2) = c_{\varsigma_2}(\tau)\omega^{2d}(\tau) = \tau\omega^{2d}(\tau).$$

Theorem 9 *Let $W \leq S_d$ be a permutation group. Let us suppose that $\sigma \in \theta W^{(2)}$, and that the pair of permutations σ, $\omega^d(\sigma)$ is braid-like. Let $\sigma^2 = \tau\omega^d(\tau)$, and let q be the order of $\tau \in W$.*

(i) The map $\theta_s \mapsto \sigma_s \bmod BW_n(\sigma)$, $s = 1, \dots, n-1$, can be extended to an isomorphism $\Sigma_n \to B_n(\sigma)/BW_n(\sigma)$;

(ii) one has $BW_n(\sigma) = A_n(\sigma)$, and $A_n(\sigma)$ is an abelian group, isomorphic to

$$\underbrace{\mathbb{Z}/(q) \coprod \cdots \coprod \mathbb{Z}/(q)}_{n-1 \text{ times}} \coprod \mathbb{Z}/(q_2) \tag{3}$$

where $q = q_2\delta$, and δ is the greatest common divisor of q and 2;

(iii) the group $B_n(\sigma)$ is an extension of the symmetric group S_n by the abelian group (3); if, in addition, q is odd, then this extension splits.

Proof: (i) According to, for example, [3, Ch. 4, Theorem 4.1], the symmetric group Σ_n has a standard presentation by generators $\theta_1, \ldots, \theta_{n-1}$, and relations

$$\theta_r\theta_s = \theta_s\theta_r,\ |r-s| \geq 2,$$

$$\theta_r\theta_s\theta_r = \theta_s\theta_r\theta_s,\ |r-s| = 1,$$

and $\theta_s^2 = 1$, $s = 1, \ldots, n-1$. Now, Proposition 6, (i), and Lemma 8, yield that $\sigma_s \bmod BW_n(\sigma)$, $s = 1, \ldots, n-1$, satisfy the above relations, so the map $\theta_s \mapsto \sigma_s \bmod BW_n(\sigma)$, $s = 1, \ldots, n-1$, can be extended to a surjective homomorphism $\Sigma_n \to B_n(\sigma)/BW_n(\sigma)$. Since σ_s, $s = 1, \ldots, n-1$, $\sigma_r\sigma_{r+1}$, $r = 1, \ldots, n-2$, $\sigma_1\sigma_3$, and σ_1^2, are $2n-1$ in number pairwise different elements $\bmod BW_n(\sigma)$, then the above homomorphism is an isomorphism

$$\Sigma_n \simeq B_n(\sigma)/BW_n(\sigma), \tag{4}$$

if $n \geq 4$. In case $n = 3$ this is true because σ_1, σ_2, and σ_1^2, are pairwise different $\bmod BW_3(\sigma)$.

(ii) By part (i), the elements of $B_n(\sigma)$, which belong to $W^{(n)}$ (that is, the elements of $B_n(\sigma)$, which are equal to the identity element $\bmod BW_n(\sigma)$), are products of conjugates of σ_s^2, and of their inverses. Lemma 8 implies that all generators of the group $BW_n(\sigma)$ are

$$\sigma_s^2,\ s = 1, \ldots, n-1, \text{ and } c_{\sigma_r}(\sigma_{r+1}^2) = \omega^{r-1}(\tau)\omega^{(r+1)d}(\tau),\ r = 1, \ldots, n-2.$$

In particular, $BW_n(\sigma) = A_n(\sigma)$.

Let $\mathbb{Z}^n$ be the free $\mathbb{Z}$-module with standard basis $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1)$. There exists a surjective homomorphism of abelian groups

$$\mathbb{Z}^n \to \langle\tau\rangle^{(n)},$$
$$(r_1, r_2, \ldots, r_n) \mapsto \tau^{r_1}\omega^d(\tau^{r_2})\ldots\omega^{(n-1)d}(\tau^{r_n}),$$

with kernel $(q)\mathbb{Z}^n$. We set

$$f_1 = e_1 + e_2,\ f_2 = e_2 + e_3, \ldots, f_{n-1} = e_{n-1} + e_n,\ f_n = e_n.$$

In terms of the basis $\{f_1, f_2, \ldots, f_{n-1}, f_n\}$, the above homomorphism can be rewritten in the form

$$\mathbb{Z}^n \to \langle\tau\rangle^{(n)}, \tag{5}$$
$$(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) \mapsto \tau^{\lambda_1}\omega^d(\tau^{\lambda_1+\lambda_2})\ldots\omega^{(n-1)d}(\tau^{\lambda_{n-1}+\lambda_n}).$$

The inverse image H_n of $A_n(\sigma)$ via (5) is the $\mathbb{Z}$-submodule of $\mathbb{Z}^n$, generated by

$$f_1, f_2, \ldots, f_{n-1}, \text{ and } g_1 = e_1 + e_3,\ g_2 = e_2 + e_4,\ \ldots,\ g_{n-2} = e_{n-2} + e_n.$$

Let us set $h_i = 2e_i$, $i = 1, \ldots, n$. We have

$$g_1 = f_1 + f_2 - h_2,\ g_2 = f_2 + f_3 - h_3, \ldots, g_{n-2} = f_{n-2} + f_{n-1} - h_{n-1},$$

and

$$h_1 + h_2 = 2f_1,\ h_2 + h_3 = 2f_2, \ldots, h_{n-1} + h_n = 2f_{n-1}.$$

Moreover, the set $\{f_1, f_2, \ldots, f_{n-1}, h_n\}$, is linearly independent over $\mathbb{Z}$. Therefore H_n is a $\mathbb{Z}$-submodule of $\mathbb{Z}^n$ with basis $\{f_1, f_2, \ldots, f_{n-1}, h_n\}$. The restriction of the homomorphism (5) on H_n has kernel $H_n \cap (q)\mathbb{Z}^n$, so we obtain an isomorphism of $\mathbb{Z}/(q)$-modules

$$\mu\colon \mathbb{Z}/(q) \coprod \cdots \coprod \mathbb{Z}/(q) \coprod \mathbb{Z}/(q_2) \to A_n(\sigma),$$

$$(\lambda_1 \bmod q, \lambda_2 \bmod q, \ldots, \lambda_{n-1} \bmod q, \lambda_n \bmod q_2) \mapsto$$
$$\tau^{\lambda_1}\omega^d(\tau^{\lambda_1+\lambda_2})\ldots\omega^{(n-1)d}(\tau^{\lambda_{n-1}+2\lambda_n}).$$

(iii) We identify the multiplicatively written group $A_n(\sigma)$ with its additively written version via μ. We also identify the factorgroup $B_n(\sigma)/A_n(\sigma)$ with the symmetric group Σ_n via the isomorphism (4), and let

$$\pi\colon B_n(\sigma) \to \Sigma_n$$

be the corresponding canonical surjective homomorphism. We have the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}/(q) \coprod \cdots \coprod \mathbb{Z}/(q) \coprod \mathbb{Z}/(q_2) \xrightarrow{\ \mu\ } B_n(\sigma) \xrightarrow{\ \pi\ } \Sigma_n \longrightarrow 1.$$

In particular, the group $B_n(\sigma)$ is an extension of the symmetric group S_n by the abelian group (3).

In accord with Theorem 3, (i), (ii), its Corollary 4, and Proposition 6, (i), if $a \in \langle\tau\rangle^{(2)}$, $a = \tau^k\omega^d(\tau^\ell)$, then the permutations $\eta_1 = \sigma a$, $\eta_2 = \omega^d(\eta_1)$, $\ldots$, $\eta_{n-1} = \omega^{(n-2)}(\eta_1)$, satisfy the braid relations, hence the group $B_n(\sigma a)$ is n-braid-like. If, $k + \ell \equiv -1 \bmod q$, then $\eta_s^2 = 1$, $s = 1, \ldots, n-1$, and in this case the map $\theta_s \mapsto \eta_s$, $s = 1, \ldots, n-1$, can be extended to a homomorphism $\rho\colon \Sigma_n \to B_n(\sigma a)$. Let us suppose, in addition, that q is odd. Then for any two integers k, ℓ, we have $a \in A_n(\sigma)$, hence $B_n(\sigma a)$ is a subgroup of $B_n(\sigma)$, and the homomorphism ρ splits π.

As an immediate consequence of the above theorem, we obtain

Corollary 10 *If the order q of the permutation τ is odd, then the abstract group $B_n(\sigma)$ does not depend on the permutation σ, but only on q.*

We set $\iota_s = c_{\sigma_s}$, $s = 1, \ldots, n-1$. Then ι_s are automorphisms of the group $\mathbb{Z}/(q) \cdots \coprod \mathbb{Z}/(q) \coprod \mathbb{Z}/(q_2)$. Taking into account Lemma 8, we have that ι_s are involutions with $\iota_s(f_r) = g_{\min\{r,s\}}$ if $|r - s| = 1$, $\iota_s(f_s) = f_s$, $\iota_s(f_r) = f_r$ if $|r - s| \geq 2$, and $\iota_{n-1}(h_n) = h_{n-1}$, for any $s = 1, \ldots, n-1$, and $r = 1, \ldots, n-2$.

Proposition 11 *The monodromy homomorphism*

$$m\colon \Sigma_n \to Aut(\mathbb{Z}/(q) \coprod \cdots \coprod \mathbb{Z}/(q) \coprod \mathbb{Z}/(q_2)),\ \theta_s \mapsto \iota_s,$$

that corresponds to the extension from Theorem 9, (iii), is injective.

Proof: Since the automorphisms ι_s, $s = 1, \ldots, n-1$, $\iota_r\iota_{r+1}$, $r = 1, \ldots, n-2$, $\iota_1^2 = id$, and $\iota_1\iota_3$, are pairwise different, the homomorphism m is injective when $n \geq 4$. The existence of three pairwise involutions ι_1, ι_2, and ι_1^2 yields that the homomorphism m is injective also for $n = 3$.

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